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Convergence of Gaussian quadrature formulas on infinite intervals

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Abstract

More general and stronger estimations of bounds for the fundamental functions of Hermite interpolation of high order on an arbitrary system of nodes on infinite intervals are given. Based on this result, convergence of Gaussian quadrature formulas for Riemann–Stieltjes integrable functions on an arbitrary system of nodes on infinite intervals is discussed.

Keywords: Fundamental polynomials; Gaussian quadrature formulas; Convergence

1. Introduction

Let $n \in \mathbb{N}$ $(n \geqslant 2)$, $m_{kn} \in \mathbb{N}$ $(k = 1, 2, \dots, n,\ n = 2, 3, \dots)$, and

$$X := \{x_{1n}, x_{2n}, \dots, x_{nn}\}, -\infty < x_{nn} < x_{n-1,n} < \cdots < x_{1n} < +\infty. \tag{1.1}$$

Throughout this paper let $N_n := \sum_{k=1}^{n} m_{kn} - 1$ and $m := \sup_{n \geqslant 2} \max_{1 \leqslant k \leqslant n} m_{kn} < +\infty$. In what follows, $m_{kn}, x_{kn}, \dots,$ will be denoted by $m_k, x_k, \dots,$ respectively. We assume that $d\alpha$ is a measure function on $\mathbf{R}$ with all moments of $d\alpha$ being finite. Denote by P_{N_n} the set of polynomials of degree at most N_n and by A_{jk} the

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fundamental polynomials for Hermite interpolation, i.e., $A_{jk} \in P_{N_n}$ satisfy

$$A_{jk}^{(p)}(x_q) = \delta_{jp}\delta_{kq}, \quad p = 0, 1, \dots, m_q - 1, \quad j = 0, 1, \dots, m_k - 1,$$
$$q, k = 1, 2, \dots, n. \tag{1.2}$$

To give an explicit formula for A_{jk} set

$$L_k(x) = \prod_{q=1, q\neq k}^{n} \left(\frac{x - x_q}{x_k - x_q} \right)^{m_q}, \quad k = 1, 2, \dots, n,$$

$$b_{vk} = \frac{1}{v!} [L_k(x)^{-1}]_{x=x_k}^{(v)}, \quad v = 0, 1, \dots, m_k - 1, \quad k = 1, 2, \dots, n, \tag{1.3}$$

$$B_{jk}(x) = \sum_{v=0}^{m_k - j - 1} b_{vk}(x - x_k)^v, \quad j = 0, 1, \dots, m_k - 1, \quad k = 1, 2, \dots, n. \tag{1.4}$$

Then we have [3]

$$A_{jk}(x) = \frac{1}{j!}(x - x_k)^j B_{jk}(x) L_k(x), \quad j = 0, 1, \dots, m_k - 1, \quad k = 1, 2, \dots, n. \tag{1.5}$$

In this paper, we need the following notations:

$$d_1 = \begin{cases} |x_1 - x_2|, & \text{for } m_2 > 1, \\ |x_1 - x_3|, & \text{for } m_2 = 1, \end{cases}$$

$$d_n = \begin{cases} |x_{n-1} - x_n|, & \text{for } m_{n-1} > 1, \\ |x_{n-2} - x_n|, & \text{for } m_{n-1} = 1, \end{cases}$$

$$d_k = \max\{|x_{k-1} - x_k|, \ |x_k - x_{k+1}|\}, \ 2 \leqslant k \leqslant n - 1,$$

$$D_n = \max_{1 \leqslant k \leqslant n} d_k, \quad n = 1, 2, \dots,$$

$$\sigma_n(x) = sgn \prod_{k=1}^{n} (x - x_k)^{m_k},$$

$$\lambda_{jkn} = \int_{-\infty}^{+\infty} A_{jk}(x)\sigma_n(x)\, d\alpha(x), \quad j = 0, 1, \dots, m_k - 1, \quad k = 1, 2, \dots, n, \tag{1.6}$$

$$Q_n(d\alpha; f) = \sum_{k=1}^{n} \lambda_{0k}(d\alpha) f(x_{kn}(d\alpha)), \quad n = 2, 3, \dots, \quad f \in S(d\alpha). \tag{1.7}$$

Here $S(d\alpha)$ stands for the set of all Riemann–Stieltjes integrable functions on **R**. For convenience of use, now we give three definitions.

Definition 1.1 (Freud [1, p. 62]). $d\alpha \in \mathscr{E}$ means that if a measure $d\beta$ satisfies

$$\int_{\mathbf{R}} x^n \, d\beta(x) = \int_{\mathbf{R}} x^n \, d\alpha(x), \quad n = 0, 1, \dots,$$

then

$$\beta(x) - \beta(-\infty) = \alpha(x) - \alpha(-\infty),$$

apart from denumerably many points of discontinuity.

Definition 1.2 (Freud [1, p. 69]). $f \in S_0(d\alpha)$ means that $f \in S(d\alpha)$ and there exist three positive numbers s, A, B such that

$$|f(x)| \leqslant A + B|x|^s, \quad x \in \mathbf{R}.$$

Definition 1.3 (Freud [1, p. 88]). Let

$$-\infty < x_{nn} < x_{n-1,n} < \cdots < x_{1n} < +\infty, \quad n = 1, 2, \ldots$$

and

$$a_{kn} \geqslant 0, \quad k = 1, 2, \ldots, n, \quad n = 1, 2, \ldots \ . \tag{1.8}$$

Let

$$Q_n(f) = \sum_{k=1}^{n} a_{kn} f(x_{kn}).$$

A positive quadrature procedure $\{Q_n\}$ is said to belong to the measure $d\alpha$ if the relation

$$\lim_{n\to\infty} Q_n(f) = \int_{\mathbf{R}} f(x)\, d\alpha(x) \tag{1.9}$$

holds for every polynomial f.

Theorem 2.1 in [3] gives a good estimation of bounds for the fundamental functions of Hermite interpolation of higher order on an arbitrary system of nodes on the interval $[-1, 1]$ (see Theorem A below), but it holds only on finite interval. The first aim of this paper is to improve the result of [3, Theorem 2.1] and to extend it to infinite intervals in Section 2 using many ideas of [3]. As applications of this result, the second aim of this paper is to give conditions for convergence of Gaussian quadrature formulas $Q_n(d\alpha; f)$ for $f \in S(d\alpha)$ under the assumption that all m_k, $1 \leqslant k \leqslant n$, $n \in \mathbb{N}$, are even, which improves the result of [1, Theorem 1.6, p.93] (see Theorem C below) on an arbitrary system of nodes on infinite intervals in Section 3. Of course, in this case $\sigma_n = 1$, a.e.

2. A basic theorem

First we state a known result.

Theorem A (Shi [3, Theorem 2.1]). *If for a fixed* n, $m_k - j$ *is odd and* $j < i < m_k$, $1 \leqslant k \leqslant n$, *then with* $c = 1$,

$$B_{jk}(x) \geqslant c \left| \frac{x - x_k}{\overline{d_k}} \right|^{i-j} |B_{ik}(x)|, \quad x \in I, \quad 1 \leqslant k \leqslant n.$$

$$|A_{jk}(x)| \geqslant c \frac{i!}{j!} \overline{d_k}^{\,j-i} |A_{ik}(x)|, \quad x \in I, \quad 1 \leqslant k \leqslant n,$$

where

$$1 = x_0 \geqslant x_1 > x_2 > \cdots > x_n \geqslant x_{n+1} = -1,$$

$$\overline{d_k} = \max\{|x_k - x_{k-1}|, \ |x_k - x_{k+1}|\}, \quad k = 1, 2, \ldots, n$$

and

$$I = \begin{cases} \mathbf{R} & \text{for } 2 \leqslant k \leqslant n-1, \\ (-\infty, 1] & \text{for } k = 1, \\ [1, +\infty) & \text{for } k = n. \end{cases}$$

Before stating the main result, we give an important lemma.

Lemma 2.1. *If* $m_1 - i$ *is even then*

$$B_{i1}(x) < 0, \quad x > x_1 + d_1. \tag{2.1}$$

Proof. Using [3, (2.32)] with $k = 1$ and $r = 2$ as well as [3, (2.25)], we obtain

$$(-1)^v b_{v1} \geqslant \frac{(-1)^{v-1}}{d_1} b_{v-1,1} > 0, \quad v = 1, 2, \ldots, m_1 - 1. \tag{2.2}$$

From (1.4) for $k = 1$, we have

$$\begin{aligned} B_{i1}(x) &= \sum_{v=0}^{m_1 - i - 1} b_{v1} (x - x_1)^v \\ &= \sum_{v=0}^{\frac{m_1 - i - 2}{2}} [b_{2v+1,1}(x - x_1) + b_{2v,1}](x - x_1)^{2v}. \end{aligned} \tag{2.3}$$

Since $x - x_1 > d_1$, by (2.2)

$$b_{2v+1,1}(x - x_1) + b_{2v,1} < b_{2v+1,1} d_1 + b_{2v,1} \leqslant 0. \tag{2.4}$$

Inequality (2.1) follows directly from (2.3) and (2.4).

The following result improves Theorem A and plays a crucial role in this paper.

Theorem 2.1. *If for a fixed* n, $m_k - j$ *is odd and* $j < i < m_k$, $1 \leqslant k \leqslant n$, *then*

$$B_{jk}(x) \geqslant \left| \frac{x - x_k}{d_k} \right|^{i-j} |B_{ik}(x)|, \quad x \in \mathbf{R}, \quad 1 \leqslant k \leqslant n. \tag{2.5}$$

$$|A_{jk}(x)| \geqslant \frac{i!}{j!} d_k^{j-i} |A_{ik}(x)|, \quad x \in \mathbf{R}, \quad 1 \leqslant k \leqslant n. \tag{2.6}$$

Proof. For $2 \leqslant k \leqslant n-1$, according to Theorem A, inequality (2.5) is obvious.

For $k = 1$, we separate two cases.

Case 1: $x \leqslant x_1 + d_1$.

Let

$$a = x_n - d_n, \quad b = x_1 + d_1.$$

Using the following linear transformation from the interval $[a, b]$ to the interval $[-1, 1]$

$$x' = \frac{2}{b-a} x - \frac{a+b}{b-a}, \tag{2.7}$$

one can get

$$x_i' = \frac{2}{b-a} x_i - \frac{a+b}{b-a}, \quad i = 0, 1, \dots, n, n+1,$$

where $x_0 := x_1 + d_1$, $x_{n+1} := x_n - d_n$.

Hence,

$$-1 = x_{n+1}' < x_n' < x_{n-1}' < \cdots < x_2' < x_1' < x_0' = 1.$$

Let

$$L_1'(x) = \prod_{q=2}^{n} \left(\frac{x - x_q'}{x_1' - x_q'} \right)^{m_q},$$

$$b_{v1}' = \frac{1}{v!} [L_1'(x)^{-1}]^{(v)}_{x = x_1'}, \quad v = 0, 1, \dots, m_1 - 1,$$

$$B_{j1}'(x) = \sum_{v=0}^{m_1 - j - 1} b_{v1}' (x - x_1')^v,$$

$$d_1' = \begin{cases} |x_1' - x_2'| & \text{for } m_2 > 1, \\ |x_1' - x_3'| & \text{for } m_2 = 1, \end{cases}$$

$$d_n' = \begin{cases} |x_{n-1}' - x_n'| & \text{for } m_{n-1} > 1, \\ |x_{n-2}' - x_n'| & \text{for } m_{n-1} = 1, \end{cases}$$

$$\overline{d_1'} = \max\{|x_1' - x_2'|, \ |x_1' - x_0'|\},$$

$$d_k' = d_k, \quad 2 \leqslant k \leqslant n-1.$$

Then

$$
\begin{aligned}
L_1(x) &= \prod_{q=2}^{n}\left(\frac{x-x_q}{x_1-x_q}\right)^{m_q}\\
&= \prod_{q=2}^{n}\left[\frac{x-\frac{b-a}{2}x_q{}'-\frac{a+b}{2}}{\frac{b-a}{2}(x_1{}'-x_q{}')}\right]^{m_q}\\
&= L_1{}'\left(\frac{2}{b-a}x-\frac{a+b}{b-a}\right)
\end{aligned}
$$

and

$$
[L_1(x)^{-1}]^{(v)}_{x=x_1} = \left(\frac{2}{b-a}\right)^{v}[L_1{}'(x)^{-1}]^{(v)}_{x=x_1{}'}, \quad v=0,1,\ldots,m_1-1,
$$

from which by (1.3) for $k=1$ it follows that

$$
\begin{aligned}
b_{v1} &= \frac{1}{v!}[L_1(x)^{-1}]^{(v)}_{x=x_1}\\
&= \frac{1}{v!}\left(\frac{2}{b-a}\right)^{v}[L_1{}'(x)^{-1}]^{(v)}_{x=x_1{}'}\\
&= \left(\frac{2}{b-a}\right)^{v} b_{v1}{}', \quad v=0,1,\ldots,m_1-1.
\end{aligned} \tag{2.8}
$$

Applying (1.4), (2.7) and (2.8)

$$
\begin{aligned}
B_{jk}(x) &= \sum_{v=0}^{m_1-j-1} b_{v1}(x-x_1)^{v}\\
&= \sum_{v=0}^{m_1-j-1}\left(\frac{2}{b-a}\right)^{v} b_{v1}{}'\left(x-\frac{b-a}{2}x_1{}'-\frac{a+b}{2}\right)^{v}\\
&= \sum_{v=0}^{m_1-j-1} b_{v1}{}'\left[\left(\frac{2}{b-a}x-\frac{a+b}{b-a}\right)-x_1{}'\right]^{v}\\
&= B_{j1}{}'\left(\frac{2}{b-a}x-\frac{a+b}{b-a}\right).
\end{aligned} \tag{2.9}
$$

If $x\leqslant b$ then $\frac{2}{b-a}x-\frac{a+b}{b-a}\leqslant 1$, according to Theorem A for $k=1$, by (2.9) we obtain

$$
\begin{aligned}
B_{j1}{}'\left(\frac{2}{b-a}x-\frac{a+b}{b-a}\right) &\geqslant (\overline{d_1{}'})^{j-i}\left|\left(\frac{2}{b-a}x-\frac{a+b}{b-a}\right)-x_1{}'\right|^{i-j}\left|B_{i1}{}'\left(\frac{2}{b-a}x-\frac{a+b}{b-a}\right)\right|\\
&= \left(\frac{b-a}{2}\overline{d_1{}'}\right)^{j-i}|x-x_1|^{i-j}\left|B_{i1}{}'\left(\frac{2}{b-a}x-\frac{a+b}{b-a}\right)\right|.
\end{aligned}
$$

However,

$$\frac{b-a}{2}\overline{d_1{}'} = \frac{b-a}{2}\max\{|x_1{}' - x_2{}'|,\ |x_1{}' - x_0{}'|\}$$
$$= \max\{|x_1 - x_2|,\ |x_1 - x_0|\}$$
$$= \max\{|x_1 - x_2|,\ d_1\}$$
$$= d_1.$$

Hence,

$$B_{j1}{}'\left(\frac{2}{b-a}x - \frac{a+b}{b-a}\right) \geqslant d_1^{j-i}|x - x_1|^{i-j}\left|B_{i1}{}'\left(\frac{2}{b-a}x - \frac{a+b}{b-a}\right)\right|. \tag{2.10}$$

Obviously from (2.9) and (2.10) it follows that

$$B_{j1}(x) \geqslant d_1^{j-i}|x - x_1|^{i-j}|B_{i1}(x)| = \left|\frac{x - x_1}{d_1}\right|^{i-j}|B_{i1}(x)|, \quad x \leqslant b,$$

i.e.

$$B_{j1}(x) \geqslant \left|\frac{x - x_1}{d_1}\right|^{i-j}|B_{i1}(x)|, \quad x \leqslant x_1 + d_1. \tag{2.11}$$

Case 2: $x > x_1 + d_1$.

In this case, we separate three subcases.

Case 2.1: $i = j + 1$.

Shi [3], proved that if $m_1 - j - 1$ is even, $B_{j+1,1}(x)$ has exactly one zero, say ξ, which must lie in $(x_1, +\infty)$ and we have

$$B_{j1}(x) > -\frac{x - x_1}{x_1 - x_2}B_{j+1,1}(x), \quad x \in \mathbf{R}, \tag{2.12}$$

$$B_{j1}(x) > \frac{x - x_1}{x_1 - x_2}B_{j+1,1}(x), \quad x \geqslant \xi, \tag{2.13}$$

$$B_{j+1,1}(x) > 0, \quad x < \xi. \tag{2.14}$$

Using (2.12)–(2.14) as well as Lemma 2.1

$$B_{j1}(x) > \left|\frac{x - x_1}{x_1 - x_2}\right||B_{j+1,1}(x)| \geqslant \left|\frac{x - x_1}{d_1}\right||B_{j+1,1}(x)|, \quad x > x_1 + d_1. \tag{2.15}$$

Case 2.2: $i = j + 2$.

Following the idea of [3], put

$$L_1^*(x) = L_1(x)\left(\frac{x - x_2}{x_1 - x_2}\right)^{-1}.$$

Thus,

$$b^*_{v1} = \frac{1}{v!}[L^*_1(x)^{-1}]^{(v)}_{x=x_1} = b_{v1} + \frac{1}{x_1 - x_2} b_{v-1,1}, \quad v \geqslant 1,$$

from which by (1.4) for $k = 1$ it is not difficult to see

$$B^*_{j+1,1}(x) = \sum_{v=0}^{m_1-j-2} b^*_{v1}(x - x_1)^v = B_{j+1,1}(x) + \frac{x - x_1}{x_1 - x_2} B_{j+2,1}(x). \tag{2.16}$$

Since $m_1 - j - 1$ is even, according to Lemma 2.1

$$B_{j+1,1}(x) < 0, \quad x > x_1 + d_1, \tag{2.17}$$

$$B^*_{j+1,1}(x) < 0, \quad x > x_1 + d_1. \tag{2.18}$$

Then by (2.16) and (2.18)

$$0 > B^*_{j+1,1}(x) = B_{j+1,1}(x) + \frac{x - x_1}{x_1 - x_2} B_{j+2,1}(x), \quad x > x_1 + d_1. \tag{2.19}$$

Applying [3, (2.24)] with $m_1 - j - 2$ being odd

$$B_{j+2,1}(x) > 0, \quad x \in \mathbf{R}. \tag{2.20}$$

Using (2.17) (2.19) and (2.20)

$$B_{j+1,1}(x) < -\frac{x - x_1}{x_1 - x_2} B_{j+2,1}(x) < 0, \quad x > x_1 + d_1.$$

Thus,

$$|B_{j+1,1}(x)| > \left|\frac{x - x_1}{x_1 - x_2}\right| |B_{j+2,1}(x)| \geqslant \left|\frac{x - x_1}{d_1}\right| |B_{j+2,1}(x)|, \quad x > x_1 + d_1. \tag{2.21}$$

Put (2.21) into (2.15), we see

$$B_{j1}(x) \geqslant \left|\frac{x - x_1}{d_1}\right|^2 |B_{j+2,1}(x)|, \quad x > x_1 + d_1. \tag{2.22}$$

Case 2.3: From the above induction, one can easily prove

$$B_{j1}(x) \geqslant \left|\frac{x - x_1}{d_1}\right|^3 |B_{j+3,1}(x)|, \quad x > x_1 + d_1,$$

$$B_{j1}(x) \geqslant \left|\frac{x - x_1}{d_1}\right|^4 |B_{j+4,1}(x)|, \quad x > x_1 + d_1,$$

$$\vdots$$

$$B_{j1}(x) \geqslant \left|\frac{x - x_1}{d_1}\right|^{i-j} |B_{i1}(x)|, \quad x > x_1 + d_1. \tag{2.23}$$

Using (2.11) and (2.23), inequality (2.5) is obvious for $k=1$. Similarly we can prove (2.5) for $k=n$. By the same argument as that of [3, (2.22)], we obtain (2.6).

3. Convergence of Gaussian quadrature formulas

First we state two theorems in [1].

Theorem B (Freud [1, Theorem 1.1, p.89]). *Let $d\alpha \in \mathscr{E}$ and $f \in S(d\alpha)$. Then for every positive quadrature procedure $\{Q_n\}$, belonging to the measure $d\alpha$, the relation*

$$\lim_{n\to\infty} Q_n(f) = \int_{\mathbf{R}} f(x)\, d\alpha(x)$$

holds.

Theorem C (Freud [1, Theorem 1.6, p. 98]). *Let $d\alpha \in \mathscr{E}$, $f \in S(d\alpha)$ and let $G(x)$ be a non-negative function, defined in $\mathbf{R}$, for which all derivatives exist in $\mathbf{R}$ and suppose that*

$$G^{(2v)}(x) \geqslant 0, \quad x \in \mathbf{R}, \quad v = 1, 2, \ldots,$$

$$\lim_{x\to+\infty} \frac{f(x)}{G(x)} = \lim_{x\to-\infty} \frac{f(x)}{G(x)} = 0$$

and $\int_{\mathbf{R}} G(x)\, d\alpha(x)$ exists; Moreover, as is customary, let $x_{kn} = x_{kn}(d\alpha)(k = 1, 2, \ldots, n)$ be the zeros of $P_n(d\alpha; f)$ of orthogonal polynomials; then

$$\lim_{n\to\infty} \bar{Q}_n(d\alpha; f) = \int_{\mathbf{R}} f(x)\, d\alpha(x),$$

where

$$\bar{Q}_n(d\alpha; f) = \sum_{k=1}^{n} \lambda_n(d\alpha; x_{kn}) f(x_{kn}(d\alpha))$$

with

$$\lambda_n(d\alpha; x_{kn}) = \int_{\mathbf{R}} l_{kn}(x)\, d\alpha(x)$$

and $l_{kn}(x)$ being the Lagrange fundamental interpolation polynomials over the points $x_{1n}, x_{2n}, \ldots, x_{nn}$. Let $x_{kn}(d\alpha)$ be the solution of the extremal problem

$$\int_{\mathbf{R}} \prod_{k=1}^{n} |x - x_k|^{m_k}\, d\alpha(x) = \min_{t_1 \geqslant \cdots \geqslant t_n} \int_{\mathbf{R}} \prod_{k=1}^{n} |x - t_k|^{m_k}\, d\alpha(x).$$

Based on the result of Theorem A, by the similar arguments to that of [4, Lemma 2] and [4, Corollary 1], we get the following two lemmas.

Lemma 3.1. *If $m_k - j$ is even and $0 \leqslant j < i \leqslant m_k - 2$, then*

$$|\lambda_{ik}(d\alpha)| \leqslant \sigma_n(x_k(d\alpha)+0) d_k^{i-j} \lambda_{jk}(d\alpha), \quad 1 \leqslant k \leqslant n. \tag{3.1}$$

Lemma 3.2. *Assume that all m_k, $1 \leqslant k \leqslant n$, $n \in \mathbb{N}$, are even then*

$$|\lambda_{jk}(d\alpha)| \leqslant d_k^j \lambda_{0k}(d\alpha) \leqslant d_k^j \int_{x_{k+1}(d\alpha)}^{x_{k-1}(d\alpha)} d\alpha(x), \quad 0 \leqslant j \leqslant m_k - 2, \quad 1 \leqslant k \leqslant n. \tag{3.2}$$

Lemma 3.3. *Let all m_k be even and $d\alpha$ be a measure. Assume that G is (N_n+1)th continuously differentiable on* **R** *and satisfies*

$$G(x) \geqslant 0, \quad G^{(N_n+1)}(x) \geqslant 0, \quad x \in \mathbf{R}. \tag{3.3}$$

Then,

$$\sum_{k=1}^{n} \sum_{j=0}^{m_k-2} \lambda_{jk}(d\alpha) G^{(j)}(x_k(d\alpha)) \leqslant \int_{\mathbf{R}} G(x)\, d\alpha(x). \tag{3.4}$$

Proof. Let $\varepsilon > 0$ be arbitrary and we consider the function $G_\varepsilon(x) = G(x) + \varepsilon x^{N_n+1}$ and the Hermite interpolation polynomial $H_{nm}(x)$ of degree at most equal to N_n of $G_\varepsilon(x)$, which is, on account of [1, Lemma 1.3, p.15] uniquely determined by the relations

$$H_{nm}^{(j)}(x_k) = G_\varepsilon^{(j)}(x_k), \quad j = 0, 1, \ldots, m_k - 1, \quad k = 1, 2, \ldots, n. \tag{3.5}$$

We show that $H_{nm}(x) \leqslant G_\varepsilon(x)$ holds for every x. If this were not the case for every x, that is to say, there exists a point y satisfying $H_{nm}(y) > G_\varepsilon(y)$; In view of $G_\varepsilon(x) \geqslant \varepsilon x^{N_n+1}$, we can get $H_{nm}(x) < G_\varepsilon(x)$ hold for sufficiently large values of x; then the difference $G_\varepsilon(x) - H_{nm}(x)$ would have at least one zero with odd multiplicity. Thus $G_\varepsilon - H_{nm}$ would have, the prescribed zeros included, at least $N_n + 2$ zeros, counted with their multiplicities. By repeated application of Role's theorem one would find a point x_0 where $G_\varepsilon^{(N_n+1)} - H_{nm}^{(N_n+1)}$ vanishes. This is, however, impossible, on account of $H_{nm}^{(N_n+1)}(x) \equiv 0$ and $G_\varepsilon^{(N_n+1)}(x_0) = G^{(N_n+1)}(x_0) + (N_n + 1)!\varepsilon > 0$. This contradiction proves our statement. On the basis of (3.5) and by [5, Theorem 2.1], we infer that

$$\sum_{k=1}^{n}\sum_{j=0}^{m_k-2}\lambda_{jk}(d\alpha)G_{\varepsilon}^{(j)}(x_k(d\alpha)) = \sum_{k=1}^{n}\sum_{j=0}^{m_k-2}\lambda_{jk}(d\alpha)H_{nm}^{(j)}(x_k(d\alpha))$$
$$= \int_{\mathbf{R}}\sigma_n(x)H_{nm}(x)\,d\alpha(x)$$
$$= \int_{\mathbf{R}}H_{nm}(x)\,d\alpha(x)$$
$$\leqslant \int_{\mathbf{R}}G_{\varepsilon}(x)\,d\alpha(x)$$
$$= \int_{\mathbf{R}}G(x)\,d\alpha(x) + \int_{\mathbf{R}}\varepsilon x^{N_n+1}\,d\alpha(x).$$

Hence the statement of the lemma follows by the limiting process $\varepsilon\to 0$.

Theorem 3.1. *Let $d\alpha\in\mathscr{E}, f\in S(d\alpha)$, and let all m_{kn}, $1\leqslant k\leqslant n$, $n\in\mathbb{N}$, be even. Assume that G has all derivatives on $\mathbf{R}$ and satisfies that*

$$G^{(2j)}(x)\geqslant 0, \quad x\in\mathbf{R}, \quad j=0,1,\ldots, \tag{3.6}$$

$$|G^{(j)}(x)|\leqslant cG(x), \quad x\in\mathbf{R}, \quad j=1,2,\ldots,m-2, \tag{3.7}$$

$$\lim_{|x|\to\infty}\frac{f(x)}{G(x)}=0, \tag{3.8}$$

and $\int_{\mathbf{R}}G(x)\,d\alpha(x)$ exists. If

$$\lim_{n\to\infty} D_n = 0 \tag{3.9}$$

then

$$\lim_{n\to\infty}\sum_{k=1}^{n}\lambda_{jk}(d\alpha)f(x_{kn}(d\alpha))=0, \quad j\geqslant 1 \tag{3.10}$$

and

$$\lim_{n\to\infty} Q_n(d\alpha;f)=\int_{\mathbf{R}}f(x)\,d\alpha(x). \tag{3.11}$$

Proof. We denote $\mathbb{N}_1=\{1,3,\ldots,m_k-3\}$, $\mathbb{N}_2=\{2,4,\ldots m_k-2\}$.

From Lemma 3.2 and (3.7)

$$\begin{aligned}\left|\sum_{k=1}^{n}\sum_{j\in\mathbb{N}_1}\lambda_{jk}(d\alpha)G^{(j)}(x_k(d\alpha))\right| &\leqslant c\sum_{k=1}^{n}\sum_{j\in\mathbb{N}_1}d_k^j\lambda_{0k}(d\alpha)G(x_k(d\alpha))\\ &\leqslant c\sum_{k=1}^{n}\sum_{j\in\mathbb{N}_1}D_n^j\lambda_{0k}(d\alpha)G(x_k(d\alpha))\\ &= c\sum_{k=1}^{n}\lambda_{0k}(d\alpha)G(x_k(d\alpha))\sum_{j\in\mathbb{N}_1}D_n^j\\ &\leqslant c\,\frac{D_n-D_n^{m-1}}{1-D_n^2}\sum_{k=1}^{n}\lambda_{0k}(d\alpha)G(x_k(d\alpha))\\ &= \alpha_n\sum_{k=1}^{n}\lambda_{0k}(d\alpha)G(x_k(d\alpha)),\end{aligned}$$

where $\alpha_n = c\frac{D_n-D_n^{m-1}}{1-D_n^2}$. Thus,

$$\sum_{k=1}^{n}\sum_{j\in\mathbb{N}_1}\lambda_{jk}(d\alpha)G^{(j)}(x_k(d\alpha))\geqslant -\alpha_n\sum_{k=1}^{n}\lambda_{0k}(d\alpha)G(x_k(d\alpha)). \tag{3.12}$$

Using Lemma 3.1 and (3.6)

$$\sum_{k=1}^{n}\sum_{j\in\mathbb{N}_2}\lambda_{jk}(d\alpha)G^{(j)}(x_k(d\alpha))\geqslant 0 \tag{3.13}$$

and

$$\sum_{k=1}^{n}\lambda_{0k}(d\alpha)G(x_k(d\alpha))\geqslant 0. \tag{3.14}$$

By Lemma 3.3 as well as (3.12)–(3.14) one can get

$$\begin{aligned}\int_{\mathbf{R}}G(x)\,d\alpha(x) &\geqslant \sum_{k=1}^{n}\sum_{j=0}^{m_k-2}\lambda_{jk}(d\alpha)G^{(j)}(x_k(d\alpha))\\ &= \sum_{k=1}^{n}\sum_{j=1}^{m_k-2}\lambda_{jk}(d\alpha)G^{(j)}(x_k(d\alpha)) + \sum_{k=1}^{n}\lambda_{0k}(d\alpha)G(x_k(d\alpha))\\ &= \sum_{k=1}^{n}\sum_{j\in\mathbb{N}_1}\lambda_{jk}(d\alpha)G^{(j)}(x_k(d\alpha)) + \sum_{k=1}^{n}\sum_{j\in\mathbb{N}_2}\lambda_{jk}(d\alpha)G^{(j)}(x_k(d\alpha))\\ &\quad + \alpha_n\sum_{k=1}^{n}\lambda_{0k}(d\alpha)G(x_k(d\alpha)) + (1-\alpha_n)\sum_{k=1}^{n}\lambda_{0k}(d\alpha)G(x_k(d\alpha))\\ &\geqslant (1-\alpha_n)\sum_{k=1}^{n}\lambda_{0k}(d\alpha)G(x_k(d\alpha)).\end{aligned} \tag{3.15}$$

Then from (3.9) and (3.15), we have

$$\limsup_{n\to\infty} \sum_{k=1}^{n} \lambda_{0k}(d\alpha)G(x_k(d\alpha)) \leqslant \int_{\mathbf{R}} G(x)\, d\alpha(x). \tag{3.16}$$

Let $\varepsilon > 0$ be arbitrary; we choose a T such that for $|x| \geqslant T$ the inequality $|f(x)| \leqslant \varepsilon G(x)$ holds and let $\alpha(x)$ be continuous at the points $\pm T$. Hence for $j \geqslant 1$ and $f \in S(d\alpha)$, by means of Lemma 3.2, we see that

$$\begin{aligned}
\left|\sum_{|x_k|\leqslant T} \lambda_{jk}(d\alpha) f(x_k(d\alpha))\right| &\leqslant \sum_{|x_k|\leqslant T} d_k^j \lambda_{0k}(d\alpha)|f(x_k(d\alpha))| \\
&\leqslant \sum_{|x_k|\leqslant T} d_k^j \int_{x_{k+1}}^{x_{k-1}} |f(x_k(d\alpha))|\, d\alpha(x) \\
&\leqslant D_n^j \sum_{|x_k|\leqslant T} |f(x_k(d\alpha))|(\alpha(x_{k-1}) - \alpha(x_{k+1})) \\
&= D_n^j \Bigg[\sum_{|x_k|\leqslant T} |f(x_k(d\alpha))|(\alpha(x_{k-1}) - \alpha(x_k)) \\
&\quad + \sum_{|x_k|\leqslant T} |f(x_k(d\alpha))|(\alpha(x_k) - \alpha(x_{k+1})) \Bigg].
\end{aligned} \tag{3.17}$$

Because $f \in S(d\alpha)$, according to [2, Chapter 3, Theorem 21], we get $|f| \in S(d\alpha)$. Thus we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \left[\sum_{|x_k|\leqslant T} |f(x_k(d\alpha))|(\alpha(x_{k-1}) - \alpha(x_k)) + \sum_{|x_k|\leqslant T} |f(x_k(d\alpha))|(\alpha(x_k) - \alpha(x_{k+1})) \right] \\
&\quad = 2 \int_{|x|\leqslant T} |f(x)|\, d\alpha(x) \\
&\quad \leqslant 2 \int_{\mathbf{R}} |f(x)|\, d\alpha(x).
\end{aligned} \tag{3.18}$$

Then by (3.9), (3.17) and (3.18)

$$\lim_{n\to\infty} \left| \sum_{|x_k|\leqslant T} \lambda_{jk}(d\alpha) f(x_k(d\alpha)) \right| = 0.$$

Using Lemma 3.2, (3.16) and noticing that $|f(x)| \leqslant \varepsilon G(x)$ for $|x| > T$, we have

$$\begin{aligned}\limsup_{n\to\infty}\left|\sum_{|x_k|>T} \lambda_{jk}(d\alpha) f(x_k(d\alpha))\right| &\leqslant \limsup_{n\to\infty} \sum_{|x_k|>T} d_k^j \lambda_{0k}(d\alpha)\varepsilon G(x_k(d\alpha))\\ &\leqslant \varepsilon \limsup_{n\to\infty} \sum_{k=1}^{n} D_n^j \lambda_{0k}(d\alpha) G(x_k(d\alpha))\\ &= \varepsilon\left[\lim_{n\to\infty} D_n^j\right] \limsup_{n\to\infty} \sum_{k=1}^{n} \lambda_{0k}(d\alpha) G(x_k(d\alpha))\\ &\leqslant \varepsilon \lim_{n\to\infty} D_n^j \int_{\mathbf{R}} G(x)\, d\alpha(x).\end{aligned}$$

Hence

$$\lim_{n\to\infty}\left|\sum_{|x_k|>T} \lambda_{jk}(d\alpha) f(x_k(d\alpha))\right| = 0.$$

However

$$\left|\sum_{k=1}^{n} \lambda_{jk}(d\alpha) f(x_k(d\alpha))\right| \leqslant \left|\sum_{|x_k|\leqslant T} \lambda_{jk}(d\alpha) f(x_k(d\alpha))\right| + \left|\sum_{|x_k|>T} \lambda_{jk}(d\alpha) f(x_k(d\alpha))\right|.$$

Thus

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{jk}(d\alpha) f(x_k(d\alpha)) = 0, \quad j \geqslant 1, \quad f \in S(d\alpha). \tag{3.19}$$

Then by means of (3.19) and Definition 1.2

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{jk}(d\alpha) f(x_k(d\alpha)) = 0, \quad j \geqslant 1, \quad f \in S_0(d\alpha). \tag{3.20}$$

From Definition 1.2, if f is a polynomial, it is easy to see that

$$f^{(j)} \in S_0(d\alpha), \quad j = 0, 1, 2, \dots \ . \tag{3.21}$$

In [5, Theorem 2.1], we have the generalized Gaussian quadrature formula

$$\int_{\mathbf{R}} f(x)\sigma_n(x)\, d\alpha(x) = \sum_{k=1}^{n} \sum_{j=0}^{m_k-2} \lambda_{jk}(d\alpha) f^{(j)}(x_k(d\alpha)), \tag{3.22}$$

which is exact for every polynomial $f \in P_{N_n}$.

It follows from (3.20) and (3.21) that for each polynomial f

$$\lim_{n\to\infty} \sum_{k=1}^{n} \sum_{j=1}^{m_k-2} \lambda_{jk}(d\alpha) f^{(j)}(x_k(d\alpha)) = 0. \tag{3.23}$$

By virtue of (1.7), (3.22) and (3.23)

$$\lim_{n\to\infty} Q_n(d\alpha;f) = \int_{\mathbf{R}} f(x)\sigma_n(x)\,d\alpha(x) = \int_{\mathbf{R}} f(x)\,d\alpha(x), \tag{3.24}$$

which is exact for every polynomial f.

According to Lemma 3.1, we have

$$\lambda_{0k}(d\alpha)\geqslant 0. \tag{3.25}$$

Then by means of Definition 1.3, (3.24) and (3.25), we get the positive quadrature procedure $\{Q_n(d\alpha;f)\}$ belonging to $d\alpha$. According to Theorem B, for $d\alpha\in\mathscr{E}$,

$$\lim_{n\to\infty} Q_n(d\alpha;f) = \int_{\mathbf{R}} f(x)\,d\alpha(x), \quad f\in S_0(d\alpha). \tag{3.26}$$

Applying (3.26) to the function, coinciding in $[-T,+T]$ with $f(x)$ but vanishing outside of $[-T,+T]$, one obtains

$$\lim_{n\to\infty} \sum_{|x_k|\leqslant T} \lambda_{0k}(d\alpha) f(x_k(d\alpha)) = \int_{-T}^{T} f(x)\,d\alpha(x). \tag{3.27}$$

By reason of (3.25), (3.16) and $|f(x)|\leqslant \varepsilon G(x)$ we obtain

$$\begin{aligned}\limsup_{n\to\infty}\left|\sum_{|x_k|>T} \lambda_{0k}(d\alpha) f(x_k(d\alpha))\right| &\leqslant \limsup_{n\to\infty} \sum_{|x_k|>T} \lambda_{0k}(d\alpha)|f(x_k(d\alpha))| \\ &\leqslant \varepsilon \limsup_{n\to\infty} \sum_{k=1}^{n} \lambda_{0k}(d\alpha) G(x_k(d\alpha)) \\ &\leqslant \varepsilon \int_{\mathbf{R}} G(x)\,d\alpha(x)\end{aligned} \tag{3.28}$$

and

$$\left|\int_{T}^{\infty} f(x)\,d\alpha(x) + \int_{-\infty}^{-T} f(x)\,d\alpha(x)\right| \leqslant \varepsilon \int_{\mathbf{R}} G(x)\,d\alpha(x). \tag{3.29}$$

Using (3.27)–(3.29) as well as the limiting process $\varepsilon\to 0$, we get

$$\lim_{n\to\infty} Q_n(d\alpha;f) = \int_{\mathbf{R}} f(x)\,d\alpha(x), \quad f\in S(d\alpha). \qquad \square$$

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